

# Bose Operators, Truncation, Lie Algebras and Spectrum

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**Abstract.** We study truncated Bose operators in finite dimensional Hilbert spaces. In particular the Lie algebra structure and the spectrum of the truncated Bose operators are discussed.

## 1 Introduction

Let  $b^\dagger, b$  be Bose creation and annihilation operators with the commutation relation  $[b, b^\dagger] = I$ , where  $I$  is the identity operator [1]. Then for the operators

$$\hat{N} = b^\dagger b, \quad b^\dagger, \quad b, \quad I$$

we find the commutators  $[b^\dagger b, b^\dagger] = b^\dagger$ ,  $[b^\dagger b, b] = -b$ ,  $[b^\dagger, b] = -I$ . All the other commutators are 0. It is well-known [2] that a non-hermitian faithful representation by  $3 \times 3$  matrices is given by

$$b^\dagger b \rightarrow M_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I \rightarrow M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$b^\dagger \rightarrow M_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad b \rightarrow M_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since for the commutators we find

$$[M_{22}, M_{23}] = M_{23}, \quad [M_{22}, M_{12}] = -M_{12}, \quad [M_{22}, M_{13}] = 0_3$$

$$[M_{23}, M_{12}] = -M_{13}, \quad [M_{23}, M_{13}] = 0_3, \quad [M_{12}, M_{13}] = 0_3.$$

Note that the matrices  $M_{13}$ ,  $M_{23}$ ,  $M_{12}$  are nonnormal.

Here we consider the four operators

$$\hat{N} = b^\dagger b, \quad b^\dagger + b, \quad b^\dagger - b, \quad I \quad (1)$$

and truncations into finite dimensional Hilbert spaces  $\mathbb{C}^n$ . The four operators  $\hat{N}$ ,  $b^\dagger + b$ ,  $b^\dagger - b$ ,  $I$  form a basis of a Lie algebra. We obtain for the nonzero commutators

$$[b^\dagger b, b^\dagger + b] = b^\dagger - b, \quad [b^\dagger b, b^\dagger - b] = b^\dagger + b, \quad [b^\dagger + b, b^\dagger - b] = 2I.$$

Obviously the identity operator  $I$  commutes with all other operators. Thus the Lie algebra generated by these operators is not semi-simple. The adjoint representation of this Lie algebra is given by

$$b^\dagger b \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b^\dagger + b \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad b^\dagger - b \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}$$

with the identity operator mapping to the  $4 \times 4$  zero matrix. Let  $|n\rangle$ ,  $|\beta\rangle$ ,  $|\zeta\rangle$  be the number states ( $n = 0, 1, \dots$ ), coherent states ( $\beta \in \mathbb{C}$ ) and squeezed states ( $\zeta \in \mathbb{C}$ ), respectively. Then we find for the operators given by (1) [3]

$$\langle n | b^\dagger b | n \rangle = n, \quad \langle \beta | b^\dagger b | \beta \rangle = \beta \beta^*, \quad \langle \zeta | b^\dagger b | \zeta \rangle = \sinh^2(|\zeta|)$$

$$\langle n | (b^\dagger + b) | n \rangle = 0, \quad \langle \beta | (b^\dagger + b) | \beta \rangle = 2\Re(\beta), \quad \langle \zeta | (b^\dagger + b) | \zeta \rangle = 0$$

$$\langle n | (b^\dagger - b) | n \rangle = 0, \quad \langle \beta | (b^\dagger - b) | \beta \rangle = -2\Im(\beta), \quad \langle \zeta | (b^\dagger - b) | \zeta \rangle = 0$$

where  $|\beta\rangle = D(\beta)|0\rangle$  and  $|\zeta\rangle = S(\zeta)|0\rangle$  with the displacement operator  $D(\beta)$  and squeezing operator  $S(\zeta)$  given by

$$D(\beta) = \exp(\beta b^\dagger - \bar{\beta} b), \quad S(\zeta) = \exp\left(-\frac{\zeta}{2}(b^\dagger)^2 + \frac{\bar{\zeta}}{2}b^2\right).$$

We study the  $n \times n$  matrices which arise in the truncation of the four operators given by (1). Since the four operators given by (1) form a basis of a Lie algebra we ask the question whether the  $n \times n$  matrices from the truncation form a basis of a Lie algebra. Furthermore we study the spectrum of the truncated operators. Coherent states in a finite-dimensional Hilbert space have been studied by Miranowicz et al [4, 5].

We mention that this set of operators given in (1) can also be considered for Fermi systems. Let  $c^\dagger, c$  be Fermi creation and annihilation operators with  $[c, c^\dagger]_+ = I$ ,  $[c, c]_+ = 0$ ,  $[c^\dagger, c^\dagger]_+ = 0$ , where  $I$  is the identity operator. Then the operators

$$\hat{N} = c^\dagger c, \quad c^\dagger + c, \quad c^\dagger - c, \quad I$$

form a basis of a Lie algebra. We obtain the nonzero commutators

$$[c^\dagger c, c^\dagger + c] = c^\dagger - c, \quad [c^\dagger c, c^\dagger - c] = c^\dagger + c, \quad [c^\dagger + c, c^\dagger - c] = 2I - 4c^\dagger c.$$

Obviously the identity operator  $I$  commutes with all other operators. So the Lie algebra is not semi-simple. A representation of these operators would be by the  $2 \times 2$  matrices

$$c^\dagger c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad c^\dagger + c \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad c^\dagger - c \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

## 2 Truncation and Lie Algebras

To find the matrix representation of  $\hat{N} = b^\dagger b$ ,  $b^\dagger + b$ ,  $b^\dagger - b$  we are applying number states  $|n\rangle$  ( $n = 0, 1, \dots$ ) with the properties

$$b^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad b |n\rangle = \sqrt{n} |n-1\rangle.$$

The number operator  $\hat{N}$  is unbounded. Since  $b^\dagger b |n\rangle = n |n\rangle$  we obtain the infinite dimensional unbounded diagonal matrix  $\text{diag}(0, 1, 2, \dots)$ . Using the number states  $|n\rangle$  we find the matrix representation of the unbounded operators  $\hat{B} = b^\dagger + b$  as

$$\hat{B} = b^\dagger + b = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Finally  $b^\dagger - b$  is given by the matrix

$$\hat{C} = b^\dagger - b = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & -\sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The identity operator  $I$  is represented by the infinite dimensional unit matrix. Now we truncate these infinite dimensional matrices. The truncation could also be found as follows. Let  $n \geq 1$  and  $\{|0\rangle, |1\rangle, \dots, |n\rangle\}$  be an orthonormal basis in  $\mathbb{C}^{n+1}$ . Note that

$$\sum_{\ell=0}^n |\ell\rangle\langle\ell| = I_{n+1}.$$

Consider the linear operators  $((n+1) \times (n+1)$  matrices)

$$b_n = \sum_{j=1}^n \sqrt{j} |j-1\rangle\langle j|, \quad b_n^\dagger = \sum_{k=1}^n \sqrt{k} |k\rangle\langle k-1|$$

with

$$b_n^\dagger + b_n = \sum_{k=1}^n \sqrt{k} (|k-1\rangle\langle k| + |k\rangle\langle k-1|).$$

Then

$$\begin{aligned} b_n b_n^\dagger &= \sum_{j=1}^n \sum_{k=1}^n \sqrt{j} \sqrt{k} |j-1\rangle\langle j|k\rangle\langle k-1| = \sum_{k=1}^n k |k-1\rangle\langle k-1| \\ b_n^\dagger b_n &= \sum_{k=1}^n \sum_{j=1}^n \sqrt{k} \sqrt{j} |k\rangle\langle k-1|j-1\rangle\langle j| = \sum_{j=1}^n j |j\rangle\langle j| \end{aligned}$$

and we obtain the commutator

$$[b_n, b_n^\dagger] = b_n b_n^\dagger - b_n^\dagger b_n = I_{n+1} - (n+1) |n\rangle\langle n|.$$

If we select the standard basis as the orthonormal basis we obtain the truncated matrices we consider in the following.

Now we truncate the infinite-dimensional matrices to  $n \times n$  matrices acting on the Hilbert space  $\mathbb{C}^n$ , where  $n \geq 2$ . For  $n = 2$  we obtain the matrices

$$N_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2$$

when we truncate the infinite dimensional unbounded matrices  $b^\dagger b$ ,  $b^\dagger + b$  and  $b^\dagger - b$ , where  $\sigma_1, \sigma_2, \sigma_3$  denote the three Pauli spin matrices. For the commutator  $[N_2, B_2]$ ,  $[N_2, C_2]$ ,  $[B_2, C_2]$  we find

$$[N_2, B_2] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = C_2 = -i\sigma_2$$

$$[N_2, C_2] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B_2 = \sigma_1$$

$$[B_2, C_2] = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2I_1 \oplus (-1) = 2\sigma_3$$

where  $I_1$  is the  $1 \times 1$  identity matrix and  $\oplus$  denotes the direct sum. For  $n = 3$  we obtain the  $3 \times 3$  matrices

$$N_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

when we truncate the infinite dimensional unbounded matrices  $b^\dagger b$ ,  $b^\dagger + b$ ,  $b^\dagger - b$ . We find the commutator  $[N_3, B_3]$ ,  $[N_3, C_3]$ ,  $[B_3, C_3]$  as

$$[N_3, B_3] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = C_3$$

$$[N_3, C_3] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} = B_3$$

$$[B_3, C_3] = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 2I_2 \oplus 2(-2)$$

where  $I_2$  is the  $2 \times 2$  identity matrix. For  $n = 4$  we obtain the  $4 \times 4$  matrices

$$N_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

when we truncate the infinite dimensional unbounded matrices  $b^\dagger b$  and  $b^\dagger + b$  to  $4 \times 4$  matrices. We find the commutators  $[N_4, B_4]$ ,  $[N_4, C_4]$ ,  $[B_4, C_4]$  as

$$[N_4, B_4] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = C_4$$

$$[N_4, C_4] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} = B_4$$

$$[B_4, C_4] = 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = 2I_3 \oplus 2(-3)$$

where  $I_3$  is the  $3 \times 3$  identity matrix. The commutators of a truncation for arbitrary  $n$  is now obvious. We find

$$[N_n, B_n] = C_n, \quad [N_n, C_n] = B_n, \quad [B_n, C_n] = 2I_{n-1} \oplus 2(-n+1)$$

where  $I_{n-1}$  is the  $(n-1) \times (n-1)$  identity matrix. Thus the commutation relations for  $[b^\dagger b, b^\dagger + b]$ ,  $[b^\dagger b, b^\dagger - b]$  are preserved for the truncation to finite dimensional matrices, whereas the commutator  $[b^\dagger + b, b^\dagger - b]$  is not preserved, i.e. we do not find 2 times the  $n \times n$  identity matrix  $I_n$ , but the direct sum of  $2I_{n-1}$  and  $2(-n+1)$

### 3 Truncation and Spectrum

It is well known that the spectrum of the unbounded operator  $b^\dagger + b$  is the whole real axis  $\mathbb{R}$  [6]. Truncating the matrix representation of the unbounded operator  $b^\dagger + b$  up the  $6 \times 6$  matrices we obtain the symmetric matrices over  $\mathbb{R}$

$$B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ 0 & 0 & 0 & \sqrt{4} & 0 \end{pmatrix}, \quad B_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & 0 \\ 0 & 0 & 0 & \sqrt{4} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix}.$$

We find the eigenvalues and eigenvectors of these matrices. Since the matrices are symmetric over the real number the eigenvalues must be real. Furthermore the sum of the eigenvalues must be 0 since the trace of the matrices is 0 and the eigenvalues are symmetric around 0 [7]. For  $B_n$  with  $n$  odd one of the eigenvalues is always 0. We order the eigenvalues from largest to smallest. For  $B_2$  we obtain the eigenvalues 1,  $-1$  with the eigenvectors

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The eigenvalues of the matrix  $B_3$  are  $\sqrt{3}$ ,  $0$ ,  $-\sqrt{3}$  with the corresponding unnormalized eigenvectors

$$\begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -1/\sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -\sqrt{3} \\ \sqrt{2} \end{pmatrix}.$$

The eigenvalues of the matrix  $B_4$  are

$$\sqrt{3+\sqrt{6}}, \quad \sqrt{3-\sqrt{6}}, \quad -\sqrt{3-\sqrt{6}}, \quad -\sqrt{3+\sqrt{6}}$$

with the corresponding unnormalized eigenvectors

$$\begin{pmatrix} \frac{1}{\sqrt{3+\sqrt{2}\sqrt{3}+3}} \\ \frac{\sqrt{2}+\sqrt{3}}{\sqrt{3+\sqrt{2}\sqrt{3}}} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{3-\sqrt{2}\sqrt{3}}} \\ \frac{\sqrt{2}-\sqrt{3}}{-\sqrt{3-\sqrt{2}\sqrt{3}}} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{-\sqrt{3-\sqrt{2}\sqrt{3}}} \\ \frac{\sqrt{2}-\sqrt{3}}{\sqrt{3-\sqrt{2}\sqrt{3}}} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{-\sqrt{3+\sqrt{2}\sqrt{3}}} \\ \frac{\sqrt{2}+\sqrt{3}}{-\sqrt{3+\sqrt{2}\sqrt{3}}} \end{pmatrix}.$$

The eigenvalues of the matrix  $B_5$  are

$$\sqrt{5+\sqrt{10}}, \quad \sqrt{5-\sqrt{10}}, \quad 0, \quad -\sqrt{5-\sqrt{10}}, \quad -\sqrt{5+\sqrt{10}}$$

with the corresponding unnormalized eigenvectors The eigenvectors of  $B_5$  are (for  $\lambda = -\sqrt{5+\sqrt{10}}, -\sqrt{5-\sqrt{10}}, 0, \sqrt{5-\sqrt{10}}, \sqrt{5+\sqrt{10}}$ )

$$\begin{pmatrix} \frac{1}{-\sqrt{5+\sqrt{10}}} \\ \frac{(4+\sqrt{10})/\sqrt{2}}{\sqrt{5+\sqrt{10}}(2+\sqrt{10})/\sqrt{6}} \\ \frac{\sqrt{\frac{2}{3}}(2+\sqrt{10})}{\sqrt{\frac{2}{3}}(2+\sqrt{10})} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{-\sqrt{5-\sqrt{10}}} \\ \frac{(4-\sqrt{10})/\sqrt{2}}{\sqrt{5-\sqrt{10}}(2-\sqrt{10})/\sqrt{6}} \\ \frac{\sqrt{\frac{2}{3}}(2-\sqrt{10})}{\sqrt{\frac{2}{3}}(2-\sqrt{10})} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -\sqrt{\frac{1}{2}} \\ 0 \\ \sqrt{\frac{3}{8}} \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{\sqrt{5-\sqrt{10}}} \\ \frac{(4-\sqrt{10})/\sqrt{2}}{\sqrt{5-\sqrt{10}}(\sqrt{10}-2)/\sqrt{6}} \\ \frac{\sqrt{\frac{2}{3}}(2-\sqrt{10})}{\sqrt{\frac{2}{3}}(2-\sqrt{10})} \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\sqrt{5+\sqrt{10}}} \\ \frac{(4+\sqrt{10})/\sqrt{2}}{\sqrt{5+\sqrt{10}}(2+\sqrt{10})/\sqrt{6}} \\ \frac{\sqrt{\frac{2}{3}}(2+\sqrt{10})}{\sqrt{\frac{2}{3}}(2+\sqrt{10})} \end{pmatrix}.$$

For  $n \geq 6$  the eigenvalues have to be found numerically. A numerical study of the case  $n = 6$  provides the six eigenvalues  $-0.61670659019259$ ,  $0.61670659019259$ ,  $-1.889175877753$ ,  $1.889175877753$ ,  $-3.3242574335521$  and  $3.3242574335521$ . Note that the eigenvalues are symmetric around 0. A numerical study indicates that for large  $n$  the largest eigenvalue grows like  $\approx 2\sqrt{n}$ . Looking at the difference between the largest and the second largest eigenvalue a numerical study indicates that for large  $n$  one finds the scaling law  $2/n^{0.185}$ .

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